

Ultraviolet-Renormalon Reexamined

A.I. Vainshtein*

*Theoretical Physics Institute, University of Minnesota,
116 Church St. SE, Minneapolis, MN 55455*

and

Budker Institute of Nuclear Physics, 630090 Novosibirsk, Russia

V.I. Zakharov†

University of Michigan, Ann Arbor, MI 48109

Abstract

We consider large-order perturbative expansions in QED and QCD. The coefficients of the expansions are known to be dominated by the so called ultraviolet (UV) renormalons which arise from inserting a chain of vacuum-polarization graphs into photonic (gluonic) lines. In large orders the contribution is associated with virtual momenta k^2 of order $Q^2 e^n$ where Q is external momentum, e is the base of natural logs and n is the order of perturbation theory considered. To evaluate the UV renormalon we develop formalism of operator product expansion (OPE) which utilizes the observation that $k^2 \gg Q^2$. When applied to the simplest graphs the formalism reproduces the known results in a compact form. In more generality, the formalism reveals the fact that the class of the renormalon-type graphs is not well defined. In particular, graphs with extra vacuum-polarization chains are not suppressed. The reason is that while inclusion of extra chains lowers the power of $\ln k^2$ their contribution is enhanced by combinatorial factors.

PACS numbers: 11.10.Gh, 11.15.Bt, 12.38.Bx

*E-mail: vainshte@vx.cis.umn.edu

†On leave of absence from Max-Planck-Institute of Physics, Munich.

1 Introduction

Large-order behavior of perturbative expansions has been studied for about forty years starting from the seminal paper by Dyson [1] (for review and further references see Refs. [2, 3]). Although a lot of insight has been gained some practical aspects of the issue in case of QCD have not been clarified so far and attracted for this reason a renewed attention recently. In particular, let us mention here discussion of possible $1/s$ power corrections at large energies \sqrt{s} , as generated through divergences of perturbative expansions [4, 5, 6, 7]. In this note ¹ we address the problems of calculation and of calculability of the so called ultraviolet renormalon [8, 9, 10, 11] which are related to these $1/s$ corrections but of more general nature.

The importance of the ultraviolet renormalon rests on the observation that it dominates large orders both in QCD and QED [8]. To be more specific we shall have in mind perturbative calculations of the famous ratio R :

$$R(s) = \frac{\sigma(e^+e^- \rightarrow \text{hadrons})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)} \quad (1)$$

as function of energy \sqrt{s} and its Euclidean counter-part $\Pi(Q^2)$,

$$-Q^2 \frac{d\Pi(Q^2)}{dQ^2} = \frac{Q^2}{12\pi^2} \int_0^\infty \frac{R(s) ds}{(s + Q^2)^2} \quad (2)$$

as function of Euclidean momentum Q . The quantity $Q^2 d\Pi(Q^2)/dQ^2$ is represented as an expansion in QCD coupling constant $\alpha_s(Q^2)$:

$$-Q^2 \frac{d\Pi(Q^2)}{dQ^2} = (\text{parton model}) \times \sum_{n=0}^{n=\infty} a_n \alpha_s^n(Q^2) \quad , \quad (3)$$

where first three coefficients have been calculated explicitly [12].

Note that the normalization of the coupling at Q^2 , as above, guarantees that there are no log factors in the expansion (3). Since we consider Q^2 to be large the coupling $\alpha_s(Q^2)$ is small. For the sake of simplicity we will illustrate our approach mostly on example of $U(1)$ model where strong interactions are mediated by an abelian vector field. Then we consider energy to be far below the position of the Landau pole so that α_1 is small. As we shall see the conclusions are of general nature and apply both to the $U(1)$ model and QCD.

Now, the generic behavior of the expansion coefficients at large n looks as

$$a_n \xrightarrow{n \rightarrow \infty} K n^\gamma \frac{n!}{S^n} \quad (4)$$

¹The letter-type version of the paper was published in the Physical Review Letters, **73**, 1207 (1994)

where K, γ and S are constants and actually there exists a variety of sources for the factorial growth (4) resulting in different S . The constant S takes the smallest absolute value in case of the ultraviolet renormalon:

$$S_{UV \text{ renorm}} = -\frac{1}{b_0} \quad (5)$$

where b_0 is the first coefficient in the corresponding β function:

$$Q^2 \frac{d}{dQ^2} (\alpha(Q^2)) = -b_0 \alpha^2(Q^2) - b_1 \alpha^3(Q^2) + \dots \quad (6)$$

Note that in QCD b_0 is positive and the series (4) is sign alternating while in $U(1)$ case b_0 is negative. The sign alternation allows for the Borel summation of the series and from this point of view there exists an important difference between QCD and $U(1)$. In this paper we are concerned mostly with calculating of a_n at large n , not on the summation of the perturbative series.

The main technical tool we are using is the operator product expansion (OPE), or the expansion in inverse powers of k^2 where k is the virtual momentum flowing through the gluonic line (see Fig. 1). The idea of using this OPE is outlined first by Parisi [10] and is based on the observation that effectively $k^2 \gg Q^2$ in case of the ultraviolet renormalon. More specifically, we shall see later that

$$(k^2)_{eff} \sim Q^2 e^n. \quad (7)$$

where n is the order of perturbative expansion and e is the base of natural logs. We will elaborate the idea of the expansion in Q^2/k^2 on explicit examples.

Our primary objective in this paper is to work out a scheme for explicit evaluation of renormalon contributions. In Sec. 2 we will address computation of the simplest renormalon-type graphs represented on Fig. 1. In $U(1)$ case such calculations have been performed in a number of papers [13, 14, 6, 15]. In particular we reproduce the results of Beneke [6] who evaluated the contribution of the renormalon chain directly, without use of OPE. What we would like to add here is a simplification of the scheme. Moreover, the generalization to the QCD case is imminent. As the next step we generalize the procedure to the case of two renormalon chains, i.e. three-loop skeleton graphs (Sec. 3). We will demonstrate that this three-loop contribution dominates over the two-loop one.

The consideration of graphs with one or two renormalon chains naturally brings us to the problem of calculability of the renormalon contribution in general (Sec. 4). By this we mean the problem of identifying the graphs which control the coefficient in front of the factorial

(see Eq. (4)). In particular, one may increase the number of renormalon chains and the question is whether this contribution is suppressed or not.²

Superficially, the more complicated graphs can be neglected since at least one extra factor of α is not accompanied in this case by a log. The use of the operator product expansion mentioned above allows to analyze the problem in a general way. The result turns unexpected: the contribution of the graphs with extra chains is *not* suppressed and the class of the “renormalon-type” graphs is *not* well defined at all.

The reason is that we have in fact two large parameters, namely, $\ln k^2/Q^2$ and the order of the perturbative expansion, n . Moreover, effectively $\ln k^2/Q^2 \sim n$, as is implied by Eq. (31). While the coefficient in front of the highest power of the log can be found in a straightforward way – and this is the essence of the renormalization group analysis – it turns to be *not* enough to evaluate the asymptotic of a_n . The other contributions lose the log factors but gain extra factors of n because of combinatorics.

2 Operator product expansion. Two-loop example.

Let us consider polarization operator $\Pi_{\mu\nu}$,

$$\Pi_{\mu\nu} = i \int dx e^{iqx} \langle 0 | T \{ j_\mu(x) j_\nu(0) | 0 \rangle = (q_\nu q_\mu - g_{\mu\nu} q^2) \Pi(Q^2), \quad Q^2 = -q^2 \quad (8)$$

of electromagnetic current j_μ in the simplified model with N_f fermionic fields

$$j_\mu = \sum_q Q_q \bar{q} \gamma_\mu q \quad (9)$$

where Q_q are the corresponding electric charges. Strong interactions are mediated by $U(1)$ gluonic field B_μ and the Lagrangian of the model is

$$L = -\frac{1}{4} B_{\mu\nu} B^{\mu\nu} + \sum_q \bar{q} \gamma^\mu i \partial_\mu q + g B^\mu j_\mu^1 + e A^\mu j_\mu \quad (10)$$

where g and e are the strong and electromagnetic couplings, respectively, and

$$j_\mu^1 = \sum_q \bar{q} \gamma_\mu q. \quad (11)$$

² In case of QCD because of the complexity of the gauge-fixing problems even the set of the graphs delegated to each type is not so well defined a priori. We shall see, however, that the technique developed allows to circumvent this problem as well.

The sum of electric charges of fermions is taken to be zero to avoid a mixing between fields A_μ and B_μ due to fermionic loops,

$$\sum_q Q_q = 0. \quad (12)$$

The simplest renormalon-type graphs are depicted in Fig. 1 and we will explain the basic features of the technique on this example. The sum of these graphs can be cast into the following form:

$$e^2 \Pi_{\mu\nu}(q) e_{(1)}^\mu e_{(2)}^\nu = -\frac{1}{2} \int \frac{d^4 k}{(2\pi)^4} \frac{g^2(k^2)}{k^2} \langle \gamma^* | T | \gamma^* \rangle \quad (13)$$

where we have made the Euclidean rotation in the integration over a gluon momentum k_μ and have introduced polarization vectors $e_{(1,2)}^\mu$ for initial and final virtual photons. The running coupling $g^2(k^2)$ sums up vacuum bubble insertions and has the form

$$g^2(k^2) = g^2(Q^2) \left(1 + b_0 \frac{g^2(Q^2)}{4\pi} \ln \frac{k^2}{Q^2} \right)^{-1} \quad (14)$$

with the first β function coefficient

$$b_0 = -\frac{N_f}{3\pi}. \quad (15)$$

The matrix element $\langle \gamma^* | T | \gamma^* \rangle$ represents the forward amplitude of gluon-photon scattering and the operator T is

$$T = \int dx e^{ikx} T \{ j_\mu^1(x), j^{1\mu}(0) \}. \quad (16)$$

The relation (13) is evident for the graphs of Fig. 1. We will discuss in Sec. 4 how it is modified for higher loops.

By assumption – to be checked a posteriori – the momentum k flowing through the gluon line is much larger than external momentum Q . Then it is logical to start by expanding in inverse powers of k . Thus we come to consider OPE for T-product of two gluonic currents j_μ^1

$$T = \int dx e^{ikx} T \{ j_\mu^1(x), j^{1\mu}(0) \} = \sum c_i(k) O_i(0), \quad (17)$$

where O_i are local operators. We will discuss this operator expansion in more detail in the next section. Here we find coefficients c_i in the tree approximation using the Schwinger background field technique [16] (for a review, see Ref. [17]). In this technique, the lines in the graph of Fig. 2 are understood as propagators in external electromagnetic and gluonic fields and the operator T takes the form:

$$T = - \sum_q \int dx \langle x | \bar{q}(X) \gamma^\mu \frac{1}{\not{p} + \not{k}} \gamma_\mu q(X) | 0 \rangle + (k \rightarrow -k). \quad (18)$$

Note that within the framework considered operators of the coordinate X_μ and of momentum p_μ are introduced. Moreover

$$[X_\mu, p_\nu] = -ig_{\mu\nu}, \quad [X_\mu, X_\nu] = 0, \quad [p_\mu, p_\nu] = 0 \quad (19)$$

The operator P_μ is defined as

$$P_\mu = p_\mu + gB_\mu + eA_\mu Q_q \quad (20)$$

and the matrix element (18) is taken over eigenstates of operator X_μ ,

$$X_\mu|x\rangle = x_\mu|x\rangle. \quad (21)$$

The construction of the OPE reduces to an expansion of Eq. (18) in powers of P_μ . The first and second terms of the expansion vanish upon averaging over directions of 4-vector k_μ and use of equations of motion, $i\not{D}q = 0$. Thus, the expansion starts from operators of dimension six,

$$T = -\frac{4}{3k^4} \sum_q \bar{q}\gamma^\mu [P^\nu, [P_\nu, P_\mu]]q + O(k^{-6}). \quad (22)$$

Substituting the commutator

$$[P_\mu, P_\nu] = igG_{\mu\nu} + ieF_{\mu\nu}Q_q \quad (23)$$

we get

$$T = \frac{4}{3k^4} \left(e\partial^\nu F_{\nu\mu} \sum_q Q_q \bar{q}\gamma^\mu q + gD^\nu G_{\nu\mu} \sum_q \bar{q}\gamma^\mu q \right) + O(k^{-6}). \quad (24)$$

The next step is to evaluate the matrix element $\langle\gamma^*|T|\gamma^*\rangle$. The part of T containing $D^\nu G_{\nu\mu}$ will contribute only on three-loop level and will be considered in the next section. As for the part of T containing $\partial^\nu F_{\nu\mu}$ it immediately factorizes into

$$\langle\gamma^*|T|\gamma^*\rangle = \frac{4}{3k^4} [\langle\gamma^*|e\partial^\nu F_{\nu\mu}|0\rangle\langle 0|j^\mu|\gamma^*\rangle + \langle\gamma^*|j^\mu|0\rangle\langle 0|e\partial^\nu F_{\nu\mu}|\gamma^*\rangle] \quad (25)$$

where j^μ is the electromagnetic current (see Eq. (9)). The matrix element of $\partial^\nu F_{\nu\mu}$ is trivial:

$$\langle 0|e\partial^\nu F_{\nu\mu}|\gamma^*\rangle = -e \left(q^2 e_\mu^{(1)} - q_\mu(qe^{(1)}) \right). \quad (26)$$

The matrix element $\langle\gamma^*|j_\mu|0\rangle$ is given by the well-known one-loop graph (see Fig. 3) and is equal to

$$\langle\gamma^*|j_\mu|0\rangle = -\frac{4eN_f\langle Q_q^2\rangle}{3}(q^2 e_\mu^{(2)} - q_\mu(qe^{(2)})) \int \frac{d^4p}{(2\pi)^4 p^4} = -\frac{eN_f\langle Q_q^2\rangle}{12\pi^2} \ln \frac{k^2}{Q^2} \cdot (q^2 e_\mu^{(2)} - q_\mu(qe^{(2)})) \quad (27)$$

where $\langle Q_q^2 \rangle$ is the averaged square of electric charge,

$$\langle Q_q^2 \rangle = \frac{1}{N_f} \sum_q Q_q^2. \quad (28)$$

Note that the integral over the fermionic loop has been evaluated with logarithmic accuracy. The upper limit of integration, $p^2 \sim k^2$, is implied by our OPE construction while the lower bound, $p^2 \sim Q^2$, arises from account in the integrand for the external momentum Q .

Substituting the result (25) for $\langle \gamma^* | T | \gamma^* \rangle$ into (13) we come to

$$\Pi(Q^2) = - \frac{N_f \langle Q_q^2 \rangle}{144\pi^4} Q^2 \int_{k^2 \sim Q^2}^{\infty} \frac{dk^2}{k^4} \ln \frac{k^2}{Q^2} \cdot g^2(k^2) \quad (29)$$

Here $Q^2 = -q^2$ and the integration over Euclidean k^2 runs over $k^2 > Q^2$. We are interested in the expansion of $\Pi(Q^2)$ in $\alpha_1(Q^2) = g^2(Q^2)/4\pi$. Expanding Eq. (14) we get

$$g^2(k^2) = 4\pi\alpha_1(Q^2) \sum_{n=0}^{\infty} \left(-b_0\alpha_1(Q^2) \right)^n \ln^n \frac{k^2}{Q^2} \quad (30)$$

where b_0 is the first coefficient in the β function. Performing the integration over k^2 in the right-hand side of Eq. (29),

$$Q^2 \int_{k^2 \sim Q^2}^{\infty} \frac{dk^2}{k^4} \ln^n \frac{k^2}{Q^2} = n! , \quad (31)$$

we arrive at the final expression for the UV renormalon contribution:

$$\Pi(Q^2) = \frac{N_f \langle Q_q^2 \rangle}{36\pi^3 b_0} \sum_{n=1}^{\infty} \left(-b_0\alpha_1(Q^2) \right)^n n! . \quad (32)$$

Differentiating this expression we get

$$-Q^2 \frac{d\Pi(Q^2)}{dQ^2} = \frac{N_f \langle Q_q^2 \rangle}{12\pi^2} \left(-\frac{1}{3\pi b_0} \right) \sum_{n=2}^{\infty} \frac{n-1}{n} n! (-b_0\alpha_1)^n. \quad (33)$$

In other words, at large n the coefficients in α_1 expansion (for the definition see Eq. (3)) are given by

$$a_n \xrightarrow{n \rightarrow \infty} -\frac{1}{3\pi b_0} (-b_0)^n \cdot n! . \quad (34)$$

This result coincides with explicit calculations of Ref. [6].

To summarize, in this section we utilized the operator product expansion to evaluate the graphs in Fig. 1 and compared the results with direct calculations. The advantage of the operator expansion is not only the compactness of the calculation but also ensuring automatically the gauge invariance so that the generalization to QCD case is straightforward

and reduces to the change in b_0 . Indeed since the operator product expansion is based on a set of gauge invariant operators the only dependence on $\ln k^2/Q^2$ arises through the use of Eq. (30) (or its two-loop generalization) and we do not need even to specify the gauge fixing or the class of graphs involved explicitly. This technical point is especially important in case of non-abelian gauge theories.

3 Comments on operator product expansion.

In this section we combine a few simple remarks on the use of OPE exemplified in the preceding section.

First, let us demonstrate that Eq. (13) which relates the polarization operator $\Pi_{\mu\nu}$ and the forward amplitude of gluon-photon scattering is valid in any order of perturbation theory. The functional dependence on the gluon propagator $D_{\alpha\beta}(x)$ is given by the following textbook formula:

$$\exp \left\{ i \int d^4x \mathcal{L}_{int} \left(\frac{\delta}{\delta \eta_\gamma(x)} \right) \right\} \exp \left\{ - \int d^4x d^4y \eta_\alpha(x) D_{\alpha\beta}(x-y) \eta_\beta(y) \right\} \Big|_{\eta_\gamma(x)=0} \quad (35)$$

where \mathcal{L}_{int} is the interaction Lagrangian in which the vector field $A_\gamma(x)$ is substituted by the functional derivative $\delta/\delta \eta_\gamma(x)$ over the source $\eta_\gamma(x)$. The source $\eta_\gamma(x)$ is substituted by zero after the differentiation. To separate out the “hard” part of the gluon propagator $D_{\alpha\beta}^k(x)$ which corresponds to the exchange by momentum k let us present $D_{\alpha\beta}(x)$ as

$$D_{\alpha\beta}(x) = D_{\alpha\beta}^k(x) + D_{\alpha\beta}^{soft}(x) . \quad (36)$$

Then expanding Eq. (45) to the first order in $D_{\alpha\beta}^k(x)$ we are proving the relation (13) in higher orders. What is crucial for the proof it is the ordering of momenta with the momentum k as the highest one.

The first comment is to note that our calculation justifies the assumption about dominance of large k^2 . Indeed, the integrals (31) over k^2 are saturated at large n by a saddle point at $k_{eff}^2 = Q^2 e^n$ (see Eq. (7)). The width of the range of k^2 which contribute to the saddle point integration is

$$\frac{k^2 - k_{eff}^2}{k_{eff}^2} \sim \frac{1}{\sqrt{n}} \quad (37)$$

It means, in particular, that the precise value of the lower limit in integrals (29), (31) is of no importance at large n .

Thus, expansion in Q^2/k^2 is fully justified. Moreover, it is clear from the calculations above that it is the dimension of operators O_i (see Eq. (17)) which is most important.

Namely, for an operator of dimension d the contribution to a_n is proportional to

$$Q^{d-4} \int_{k^2 \sim Q^2}^{\infty} \frac{dk^2}{k^{d-2}} \ln^n \frac{k^2}{Q^2} = \frac{n!}{((d-2)/2)^{n+1}}. \quad (38)$$

In the preceding section we dealt with operators of dimension $d = 6$. Operators of higher dimensions give rise contributions which are suppressed by powers of $(d-4)/2$ and can be safely neglected.

However, the question naturally arises on the role of operators of dimension four. Although such operators did not appear in the tree approximation of the preceding section they show up via loop corrections. It is worth emphasizing therefore that these operators are not relevant to asymptotic of a_n .

Indeed, matrix elements of $d = 4$ operators over virtual photons have the form

$$\langle \gamma^* | O_i^{d=4} | \gamma^* \rangle = A_i e^2 \left(q^2 (e^{(1)} e^{(2)}) - (q e^{(1)}) (q e^{(2)}) \right) \quad (39)$$

where A_i are dimensionless and the corresponding contributions to $\Pi(Q^2)$ are

$$\Delta_i \Pi \sim \int dk^2 A_i c_i^{d=4}(k^2) g^2(k^2) \quad (40)$$

where $c_i^{d=4}$ are OPE coefficients, $c_i^{d=4} \sim 1/k^2$. To get rid of the apparent ultraviolet divergence in (40) one needs to consider instead of $\Pi(Q^2)$ the quantity $Q^2 \cdot d\Pi(Q^2)/dQ^2$ which contains no ultraviolet uncertainty.

The finiteness of $Q^2 \cdot d\Pi(Q^2)/dQ^2$ implies that $\sum_i c_i^{d=4} A_i$ is independent of Q^2 . Notice that this assertion is valid in spite of nonzero anomalous dimensions of some $d = 4$ operators. It also persists in the presence of fermionic mass terms $m_q \bar{q} q$ which contains $d = 3$ operators. The Q^2 independence $\sum_i c_i^{d=4} A_i$ in any order of perturbation theory means that the sum of contributions (40) drops off from $Q^2 \cdot d\Pi(Q^2)/dQ^2$.

Thus $d = 4$ operators do not contribute to UV renormalons and the UV renormalon calculus deals with $d = 6$ operators. In the preceding section we had one such operator explicitly:

$$O_{Fj} = e \partial^\nu F_{\nu\mu} j^\mu \quad (41)$$

where j_μ is defined by Eq. (9). The calculation of matrix element $\langle \gamma^* | O_{Fj} | \gamma^* \rangle$ (see Eq. (25)) can be interpreted as a mixing of O_{Fj} with operator

$$O_{F^2} = (e \partial^\nu F_{\nu\mu})^2, \quad (42)$$

i.e. with logarithmic accuracy we have

$$O_{Fj}|_{k^2} = O_{Fj}|_{Q^2} + \frac{N_f \langle Q_q^2 \rangle}{12\pi^2} \ln \frac{k^2}{Q^2} \cdot O_{F^2}|_{Q^2} \quad (43)$$

where we marked the normalization points of the operators. Since the diagonal anomalous dimensions of the operators O_{Fj}, O_{F^2} are vanishing Eq. (43) specifies the matrix of the anomalous dimensions completely.

Note also that the log in the right-hand side of Eq. (43) results in an enhancement of a_n by factor of n (see Eq. (31)) and this enhancement is crucial for the consistency of our calculation. The point is that in one-loop order the operator O_{F^2} appears in the OPE (17) not only through its mixing with O_{Fj} as in the tree approximation but directly as well. In that case large momentum k flows through all fermionic lines in graph of Fig. 1. However then there is no log factor similar to the one in Eq. (43) so that the direct O_{F^2} coefficient can be neglected with $1/n$ accuracy. There are also $d = 6$ four-fermionic operators which contribute to $\Pi(Q^2)$ on three-loop level and these will be discussed in the next section. Here we would like only to note that their logarithmic mixing with the operators O_{Fj} and O_{F^2} results in an enhancement of the three-loop skeleton graph.

Finally we note that so far we considered for simplicity one-loop β function for α_1 . Accounting for the second coefficient in the β function brings a non-vanishing constant γ in Eq. (4). The procedure is rather standard and we will give examples in the next section. As for the overall normalization (constant K in Eq. (4)) it is actually scheme dependent, for further discussion see [18, 19]. The constant K can be fixed in large N_f limit, which simplifies calculations greatly (see, e.g., [6, 15, 7]). As we shall see later, however, the limits of n and N_f tending to infinity are not necessarily consistent with each other. Here we keep track of large n dependence.

To summarize, the operator product expansion provides with systematic means of calculating asymptotic of the perturbative expansion coefficients a_n associated with renormalon-type graphs.

4 Three-loop example.

In this section we evaluate the renormalon contribution associated with the graphs with two chains of vacuum-polarization insertions, or three-loop skeleton graphs, see examples in Fig. 4. In fact we have already mentioned that the contribution of the operator $D^\nu G_{\nu\mu}$ (see Eq. (27)) arises only on a three-loop level and now we come to consider three-loop calculations in more detail.

Note that $D^\nu G_{\nu\mu}$ can be substituted by $(-g) \sum \bar{q} \gamma_\mu q$ via equations of motion so that we

get for the corresponding part T_1 of operator T defined by Eq. (16)

$$T_1 = -\frac{4g^2}{3k^4}O_1, \quad O_1 = \left(\sum_q \bar{q}\gamma_\mu q\right)^2. \quad (44)$$

Diagrammatically the appearance of T_1 in three-loop graphs for the polarization operator is illustrated by the diagram *b* in Fig. 4. The dotted box in this diagram presents a “penguin” mechanism for T_1 similar to the one in weak interactions [20].

A different mechanism leading to four-fermion operators is due to the exchange by two gluons, see the diagram *a* in Fig. 4. The part of the graph which corresponds to the operator T is marked by the dotted box in Fig. 4a and presented separately in Fig. 5. As far as explicit calculations of the graphs are concerned they are very similar to those used in derivation of QCD sum rules [21] since OPE is exploited in the both cases. The difference is that in the former case it is the momentum carried by the electromagnetic current which is considered to be large while now the large momentum is brought in by the gluonic current.

The subtle point here is that the relation between the scattering diagram of Fig. 5 and the box part of diagram *a* in Fig. 4 contains a combinatorial coefficient $1/2$ which is absent for the “penguin” box of Fig. 4b. Indeed, the functional dependence on the gluon propagator $D_{\alpha\beta}(x)$ is given by the following textbook formula:

$$\exp\left\{i\int d^4x \mathcal{L}_{int}\left(\frac{\delta}{\delta\eta_\gamma(x)}\right)\right\} \exp\left\{-\int d^4x d^4y \eta_\alpha(x) D_{\alpha\beta}(x-y) \eta_\beta(y)\right\}\Big|_{\eta_\gamma(x)=0} \quad (45)$$

where \mathcal{L}_{int} is the interaction Lagrangian in which the vector field $A_\gamma(x)$ is substituted by the functional derivative $\delta/\delta\eta_\gamma(x)$ over the source $\eta_\gamma(x)$. The source $\eta_\gamma(x)$ is substituted by zero after the differentiation. To separate out the “hard” part of the gluon propagator $D_{\alpha\beta}^k(x)$ which corresponds to the exchange by momentum k let us present $D_{\alpha\beta}(x)$ as

$$D_{\alpha\beta}(x) = D_{\alpha\beta}^k(x) + D_{\alpha\beta}^{soft}(x). \quad (46)$$

Then expanding Eq. (45) to the first order in $D_{\alpha\beta}^k(x)$ we get the relation (13). To maintain the same relation (13) for terms of the second order in $D_{\alpha\beta}^k(x)$ the corresponding part of gluon scattering amplitude should be taken with the extra factor $1/2!$.

Accounting for the modification described we get the following result for four-fermion operators generated by graphs of the type given by Fig. 5:

$$T_2 = -\frac{3g^2}{k^4}O_2, \quad O_2 = \left(\sum_q \bar{q}\gamma_\mu \gamma_5 q\right)^2. \quad (47)$$

Dotted boxes in Fig. 4 present are reduced to local operators for fermionic momenta much less than k . We also account for the graphs which gives logarithmical dressing of the boxes,

i.e. $(\alpha \ln(k^2/Q^2))^n$ corrections. These corrections are due to the range of momenta between k and Q and diagrammatically are graphs presenting gluon exchanges between legs of effective four-fermionic operators as well as penguin-type fermionic loops for these operators. They were absent in two-loop considerations because of vanishing anomalous dimension of the relevant operator O_{Fj} (see Eq. (41)). Operators $O_{1,2}$ have nonzero anomalous dimension.

To sum up logarithmic corrections for the matrix element $\langle \gamma^* | T | \gamma^* \rangle$ we apply the standard machinery of the renormalization group. The operator basis of the problem consists of operators O_1 , O_2 , O_{Fj} , O_{F^2} (see Eqs. (44, 47, 41, 42)). Their coefficients c_1 , c_2 , c_{Fj} , c_{F^2} are functions of μ^2 where μ is the normalization point.

The set of renormalization group equations in one-loop approximation looks as

$$\begin{aligned} \mu^2 \frac{d}{d\mu^2} c_{F^2} &= -\frac{N_f \langle Q_q^2 \rangle}{12\pi^2} c_{Fj} , \\ \mu^2 \frac{d}{d\mu^2} c_{Fj} &= -\frac{1}{12\pi^2} (c_1 + c_2) , \\ \mu^2 \frac{d}{d\mu^2} c_1 &= \frac{\alpha_1}{\pi} \left(\frac{2N_f + 1}{3} c_1 + \frac{11}{6} c_2 \right) , \\ \mu^2 \frac{d}{d\mu^2} c_2 &= \frac{3\alpha_1}{2\pi} c_1 . \end{aligned} \tag{48}$$

The solution for c_1 , c_2 has the form

$$\begin{aligned} c_1(\mu^2) &= \frac{1}{1 + (11/4\pi^2 b_0^2 \gamma_1^2)} \left\{ \left[c_1(\mu_0^2) + \frac{11}{6\pi b_0 \gamma_1} c_2(\mu_0^2) \right] \left[\frac{\alpha_1(\mu_0^2)}{\alpha_1(\mu^2)} \right]^{\gamma_1} + \right. \\ &\quad \left. \left[\frac{11}{4\pi^2 b_0^2 \gamma_1^2} c_1(\mu_0^2) - \frac{11}{6\pi b_0 \gamma_1} c_2(\mu_0^2) \right] \left[\frac{\alpha_1(\mu_0^2)}{\alpha_1(\mu^2)} \right]^{\gamma_2} \right\} \end{aligned} \tag{49}$$

$$\begin{aligned} c_2(\mu^2) &= \frac{1}{1 + (11/4\pi^2 b_0^2 \gamma_1^2)} \left\{ \left[\frac{3}{2\pi b_0 \gamma_1} c_1(\mu_0^2) + \frac{11}{4\pi^2 b_0^2 \gamma_1^2} c_2(\mu_0^2) \right] \left[\frac{\alpha_1(\mu_0^2)}{\alpha_1(\mu^2)} \right]^{\gamma_1} + \right. \\ &\quad \left. \left[-\frac{3}{2\pi b_0 \gamma_1} c_1(\mu_0^2) + c_2(\mu_0^2) \right] \left[\frac{\alpha_1(\mu_0^2)}{\alpha_1(\mu^2)} \right]^{\gamma_2} \right\} , \end{aligned} \tag{50}$$

where anomalous dimensions $\gamma_{1,2}$ are

$$\gamma_{1,2} = \frac{1}{\pi b_0} \left(\frac{2N_f + 1}{6} \pm \sqrt{\left(\frac{2N_f + 1}{6} \right)^2 + \frac{11}{4}} \right) . \tag{51}$$

Coefficients c_{Fj} and c_{F^2} are obtained then by a simple integration. Initial conditions for $c_i(\mu^2)$ are set by Eqs. (44, 47) at $\mu^2 = k^2$,

$$c_1(k^2) = -\frac{4}{3} \frac{g^2(k^2)}{k^4} , \quad c_2(k^2) = -3 \frac{g^2(k^2)}{k^4} , \quad c_{Fj}(k^2) = c_{F^2}(k^2) = 0 . \tag{52}$$

We need to calculate the value of $c_{F^2}(\mu^2 = Q^2)$ because with logarithmic accuracy the matrix element $\langle \gamma^* | T | \gamma^* \rangle$ is given by

$$\langle \gamma^* | T | \gamma^* \rangle = 2 e^2 c_{F^2}(\mu^2 = Q^2) q^2 \left[q^2 (e^{(1)} e^{(2)}) - (q e^{(1)}) (q e^{(2)}) \right] \quad (53)$$

The result for $c_{F^2}(\mu^2 = Q^2)$ is rather lengthy expression,

$$\begin{aligned} c_{F^2}(\mu^2 = Q^2) = & -\frac{1}{9} N_f \langle Q_q^2 \rangle \frac{1}{k^4} \frac{1}{g^2(k^2)} \frac{1}{\pi^2 b_0^2 + (11/4 \gamma_1^2)} \times \\ & \left\{ \left(\frac{4}{3} + \frac{15}{2\pi b_0 \gamma_1} + \frac{33}{4\pi^2 b_0^2 \gamma_1^2} \right) \frac{1}{1 + \gamma_1} \left[1 - \kappa - \frac{1}{2 + \gamma_1} (1 - \kappa^{(2+\gamma_1)}) \right] \right. \\ & \left. + \left(3 - \frac{15}{2\pi b_0 \gamma_1} + \frac{11}{3\pi^2 b_0^2 \gamma_1^2} \right) \frac{1}{1 + \gamma_2} \left[1 - \kappa - \frac{1}{2 + \gamma_2} (1 - \kappa^{(2+\gamma_2)}) \right] \right\} \end{aligned} \quad (54)$$

where

$$\kappa = \frac{\alpha_1(k^2)}{\alpha_1(Q^2)} = \left[1 + b_0 \alpha_1(Q^2) \ln \frac{k^2}{Q^2} \right]^{-1}. \quad (55)$$

The polarization operator Π is given then by integration over k ,

$$\Pi = -\frac{Q^2}{16\pi^2} \int_{Q^2}^{\infty} dk^2 g^2(k^2) c_{F^2}(\mu^2 = Q^2). \quad (56)$$

Coefficients of expansion in α_1 are defined by integrals of the type

$$Q^2 \int_{Q^2}^{\infty} \frac{dk^2}{k^4} [\alpha_1(k^2)]^\delta = [\alpha_1(Q^2)]^\delta \sum_{n=0}^{\infty} \frac{\Gamma(n + \delta + \delta_1)}{\Gamma(\delta + \delta_1)} [-b_0 \alpha_1(Q^2)]^n \quad (57)$$

where δ_1 arises from account of the second coefficient in the β function b_1 (see, e.g., [24]) and equals to

$$\delta_1 = -\frac{b_1}{b_0^2}. \quad (58)$$

The final formula for the expansion of Π in powers of $\alpha_1(Q^2)$ has the form:

$$\begin{aligned} \Pi = & \frac{N_f \langle Q_q^2 \rangle}{144\pi^2} \frac{1}{\pi^2 b_0^2 + (11/4 \gamma_1^2)} \sum_{n=2}^{\infty} [-b_0 \alpha_1(Q^2)]^n \times \\ & \left\{ \left(\frac{4}{3} + \frac{15}{2\pi b_0 \gamma_1} + \frac{33}{4\pi^2 b_0^2 \gamma_1^2} \right) \frac{1}{1 + \gamma_1} \left[\frac{\Gamma(n + 2 + \gamma_1 + \delta_1)}{\Gamma(3 + \gamma_1 + \delta_1)} - \Gamma(n + 1 + \delta_1) \right] + \right. \\ & \left. \left(3 - \frac{15}{2\pi b_0 \gamma_1} + \frac{11}{3\pi^2 b_0^2 \gamma_1^2} \right) \frac{1}{1 + \gamma_2} \left[\frac{\Gamma(n + 2 + \gamma_2 + \delta_1)}{\Gamma(3 + \gamma_2 + \delta_1)} - \Gamma(n + 1 + \delta_1) \right] \right\} \end{aligned} \quad (59)$$

Assuming for the moment $\gamma_1 = \gamma_2 = \delta_1 = 0$, i.e. omitting effects of anomalous dimensions and two-loop β function, we get for coefficients a_n defined by Eq. (3)

$$a_n \xrightarrow{n \rightarrow \infty} = -\frac{13}{36\pi^2 b_0^2} (-b_0)^n (n+1)! \quad (60)$$

Comparing this result with the Eq. (34) we see that the contribution of the four-fermion operators, or of the three-loop graphs, dominates at large n over that of the two-loop graph, it contains an extra factor n . Moreover, this conclusion is not modified if nonzero γ_i and δ_1 are accounted for. Indeed, the dependence on δ_1 is universal for two-loop and three-loop contributions and drops off from the ratio of a_n . As for the anomalous dimensions γ_i of the four-fermionic operators (see Eq. (51)), γ_1 is negative, γ_2 is positive. Notice that in the realistic case of QCD one also gets quite a few different diagonal operators both with positive and negative anomalous dimensions [21, 22]. In the large n limit the operator with the most positive γ dominates so that account for the anomalous dimensions only strengthens the conclusion on the dominance of three-loop graphs.

In the particular case when the number of flavors N_f is large, $N_f \gg 1$, one arrives at $\gamma_1 = -2$, $\gamma_2 = 0$, $\delta_1 = 0$ and Eq. (59) leads to

$$a_n \xrightarrow{n \rightarrow \infty} = -\frac{9}{8N_f^2} (-b_0)^n (n+1)! . \quad (61)$$

Although in large N_f limit the three-loop contribution is suppressed by an extra factor $1/N_f$ as compared with the two-loop one it has an extra factor n .

As an example of advantage of OPE technique used above let us mention that the summation automatically accounts for rather complicated graphs, in particular, those where the high momentum gluon propagator is inside of fermionic bubble for a low ($\sim Q$) momentum gluon.

To summarize, we have demonstrated in this section that three-loop graphs generate new, four-fermionic operators. The technique of the operator product expansion works the same simple as in the two-loop case. Although the final expressions accounting for the anomalous dimensions are somewhat cumbersome they are straightforward. The most non-trivial conclusion is that four-fermionic operators dominate at large n .

5 Calculability of ultraviolet renormalon. Conclusions.

In this section we comment on the definition of the ultraviolet renormalon. The basic observation is that if one tries to identify the ultraviolet renormalon as a set of graphs producing a certain asymptotic behavior, $a_n \sim n!(b_0)^n$, then this set of graphs is in fact ill-defined.

The demonstration of this point is straightforward. Let us go back to our operator product expansion (17). The coefficient functions are represented as series in $\alpha_1(k^2)$

$$c_i(k^2) = h_0 + h_1\alpha_1(k^2) + h_2\alpha_1^2(k^2) + \dots + h_l\alpha_1^l(k^2) + \dots \quad (62)$$

The standard and tacit assumption is that one can approximate $c_i(k^2)$ by, say, first term in this expansion as far as $\alpha_1(k^2)$ is small. This logic does not work however if we use the operator product expansion (17) to evaluate the asymptotic of the coefficients a_n of expansion in our “original” $\alpha_s(Q^2)$. Indeed, from Eq. (57) we conclude that for any finite l the contribution of h_l to the asymptotic of a_n has the same n dependence. This means, in turn, that all the terms in the expansion (62) are the same important.

Technically this happens because originally there are two large parameters, n and $\ln(k^2/Q^2)$, and only the logs are controlled by the renormalization group. As a result of integration over k^2 powers of logs are converted into powers of n (see, e.g., Eq. (31)) and the two parameters merge. The reason why all h_l above contribute the same order of magnitude to asymptotical a_n is that lower powers of logs have larger statistical weights. This can be traced directly by inspecting combinatorics of l renormalon chains of graphs. As a result the two large parameters, combinatorial n and $\ln k^2$, get mixed up and the actual asymptotic of a_n is determined by an extremum in the two-dimensional parameter space. Finding this extremum looks much harder a problem than fixing the leading logs. It is not ruled out, in particular, that the contributions of various h_l in Eq. (62) cancel between themselves and the true asymptotic is very different from (4), (5).

It is worth emphasizing that this difficulty with obtaining the asymptotic of the expansion coefficients is specific for the ultraviolet renormalon and does not plague the calculation of the infrared renormalon. The reason for this looks pure technical. On the other hand, renormalons are reflections of the Landau poles in the running couplings (see, e.g., [2]) and if indeed the true asymptotic of the UV renormalon is different from (4),(5) then the Landau pole in ultraviolet does not manifest itself in, say, polarization operator $\Pi(Q^2)$. Thus there is a remarkable difference in the statuses of the infrared and ultraviolet renormalons.

As another illustration of the uncertainties in the evaluation of the ultraviolet region in perturbation theory let us consider a tower of renormalons which means that we introduce another scale, k' such that

$$(k')^2 \gg k^2 \gg Q^2, \quad (k')^2 \sim k^2 e^l \quad (63)$$

and l is large enough. Then coefficients h_l are given by

$$h_l \sim l! (-b_0)^l. \quad (64)$$

The corresponding contribution of each h_l to the asymptotic of a_n is again

$$a_n \xrightarrow{n \rightarrow \infty} \text{const} \cdot n! (-b_0)^n. \quad (65)$$

Moreover, we ignored the anomalous dimension of the operators which arise at the scale k' so that the true asymptotic can again be enhanced. But even these simple estimates suffice to demonstrate that no matter how large the virtual momenta are they do not detach from the asymptotic of a_n .

Thus, we conclude that at present there are no technical means to evaluate the true asymptotic of the UV renormalon.

As for possible phenomenological implications of the results obtained we have unfortunately little to say. Pessimistically, one may argue that all the problems with evaluating the UV asymptotic are not physical and the reason why we fail is simply because we want to represent as a series in $\alpha_s(Q^2)$ the coupling $\alpha_s(k^2)$ which is much smaller than $\alpha_s(Q^2)$ at $k^2 \gg Q^2$. To this end we need to keep many sign alternating terms absolute value of which is much larger than $\alpha_s(k^2)$ itself. If we do not endeavor the expansion, however, the whole UV contribution is small and uninteresting.

On the optimistic side, one might say that the divergences of the perturbative expansions reveal power like corrections and the failure to evaluate the UV renormalon signifies that the corresponding $1/Q^2$ terms are important. Indeed, applying the Borel summation one can convert the UV renormalon series into $1/Q^2$ term:

$$K \sum_{N \text{ large}}^{\infty} \alpha_s^n(Q^2) (-1)^n (b_0)^n n! \longrightarrow \frac{K}{Q^2} \quad (66)$$

and if the constant K happens to be large numerically then the Q^{-2} correction is also large. Moreover, one can generate the Q^{-2} terms without invoking the Borel summation either by varying the normalization point of α_s in the expansion [23] or by exploiting the conformal mapping of perturbative expansions [3].

One general remark concerning the power like corrections is now in order. Generally speaking, one would not expect these corrections be important numerically since they are naturally screened by lower order perturbative terms. The examples of successful phenomenological applications are known so far for IR renormalons only. For example, the first IR renormalon is associated with Q^{-4} term [24]. Phenomenologically, the corresponding correction is described by the so called gluon condensate [21], $\langle 0 | \alpha_s (G_{\mu\nu}^a)^2 | 0 \rangle$ where $G_{\mu\nu}^a$ is the gluonic field strength tensor. The basic assumption is that this condensate is large and exceeds at moderate Q^2 even the lowest terms of the perturbative expansion. Having only perturbative expansions in hand, this assumption is very difficult to justify.

Similarly, to have the UV renormalon important we need to assume that the constant K is large. Better to say, we need a mismatch between large K in the asymptotic and numerical values of the lowest a_n . The complicated structure of the UV renormalon revealed above

may be considered as suggesting such a complicated structure of the perturbative series. The main problem is not that such an assumption would be too bizarre but rather lack of a framework which would allow to develop a phenomenology basing on this assumption. Indeed, in case of the Q^{-4} terms the main point was the possibility to relate these terms in various channels, that is not possible at present in case of the UV renormalon.

Although we do not have constructive ways to introduce this phenomenology let us note that the natural direction would be Nambu-Jona-Lasinio type models [25] since in these models one exploits four-fermion interaction as a low-energy effective interaction in QCD. We have seen that UV renormalon is naturally related to four-fermion interaction as well. One could hope to match $1/Q^2$ corrections due to the renormalon with NJL type phenomenology.

6 Acknowledgments

The authors are grateful to M. Beneke, V. Gribov, G. Grunberg and M. Shifman for useful discussions. We are particularly thankful to M. Beneke, the comparison with his unpublished results helped us to localize some mistakes in the preprint version of the paper. This work was supported in part by DOE under the grant number DE-FG02-94ER40823.

References

- [1] F.J. Dyson, Phys. Rev. **85**, 861 (1952).
- [2] *Large-Order Behavior of Perturbation Theory*, edited by J.C. Gillou and J. Zinn-Justin, Current Physics - Sources and Comments Vol. 7 (North-Holland, Amsterdam, 1990).
- [3] A.H. Mueller, in *QCD - Twenty Years Later* (World Scientific, Singapore, 1993).
- [4] V.I. Zakharov, Nucl. Phys. **B385**, 452 (1992).
- [5] Lowell S. Brown and Laurence G. Yaffe, Phys. Rev. **D45**, R398 (1992); Lowell S. Brown, Laurence Yaffe, and Chengxing Zhai, Phys. Rev. **D46**, 4712 (1992).
- [6] M. Beneke, Nucl.Phys. **B405**, 424 (1993).
- [7] C.N. Lovett-Turner and C.J. Maxwell, Nucl.Phys. **B432** 147 (1994).
- [8] G. 't Hooft, in *The Whys of Subnuclear Physics*, Erice Lecture Notes 1977, edited by A. Zichichi (Plenum, New York, 1978).

- [9] B. Lautrup, Phys. Lett. **B69**, 109 (1978).
- [10] G.Parisi, Phys. Lett. **B76**, 65 (1977).
- [11] M.C. Bergere and F. David, Phys. Lett. **B315**, 412 (1984).
- [12] S.G. Gorishny, A.L. Kataev, and S.A. Larin, Phys. Lett. **B259**, 144 (1991);
L.R. Surguladze and M.A. Samuel, Phys. Rev. Lett. **66**, 560 (1991).
- [13] R. Coquereaux, Phys. Rev. **D23**, 2276 (1981).
- [14] H. Kawai, T.Kinoshita, and Y. Okamoto, Phys. Lett. **B260**, 193 (1991).
- [15] D.J. Broadhurst, Zeit. Phys. **C58**, 339 (1993).
- [16] J. Schwinger, Phys.Rev. **82**, 664 (1951); *Particles, Sources and Fields* Vols. 1,2
(Addison-Wesley, 1969).
- [17] V.A. Novikov, M.A. Shifman, A.I. Vainshtein, and V.I. Zakharov, Fortschritte der
Physik **32**, 582 (1984).
- [18] M. Beneke and V.I. Zakharov, Phys. Lett. **B312**, 340 (1993)
- [19] G. Grunberg, Phys. Lett. **B304**, 183 (1993).
- [20] A.I. Vainshtein, V.I. Zakharov and M.A. Shifman. JETP Lett. **22**, 123 (1975);
Sov.Phys.JETP **45**, 670 (1977); Nucl.Phys. **B120** 316 (1977)
- [21] M.A. Shifman, A.I. Vainshtein, and V.I. Zakharov, Nucl. Phys. **B147**, 448 (1979).
- [22] D.T. Barfoot and D.J. Broadhurst, Phys. Lett. **B166**, 347 (1986).
- [23] M. Beneke and V.I. Zakharov, Phys. Rev. Lett. **69**, 2472 (1992).
- [24] A.H. Mueller, Nucl. Phys. **B250**, 327 (1984).
- [25] Y. Nambu and G. Jona-Lasinio, Phys Rev. **122**, 345 (1961); Phys Rev. **124**, 246 (1961).

Figure captions.

Fig 1. Building up the simplest renormalon-type graph. Dashed line denotes gluon while solid lines refer to fermions. One starts with an exchange of a vector particle of momentum k and inserts vacuum polarization bubbles n times.

Fig 2. The graph used to evaluate coefficients of operator expansion associated with the simplest renormalon graphs of Fig. 1. Momentum k carried by the gluonic line is considered to be large. The fermion is understood to propagate in external electromagnetic and gluonic fields so that the graph is in fact a subgraph of the one-gluon exchange depicted in Fig. 1.

Fig 3. One-loop graph describing transition of electromagnetic current j_μ into a virtual photon with momentum q .

Fig 4. Three-loop skeleton graphs giving rise to four-fermionic operators. Momentum k flowing through the gluonic lines is considered to be large so that the operator product expansion is an expansion in inverse powers of k^{-2} . The dotted boxes mark subgraphs producing four fermionic operators O_2 and O_1 (see Eqs. (47) and (44), respectively).

Fig 5. The graph giving rise to the operator O_2 in the limit of large k . It is a subgraph of the first graph in Fig. 4 where it is marked by a dotted box.

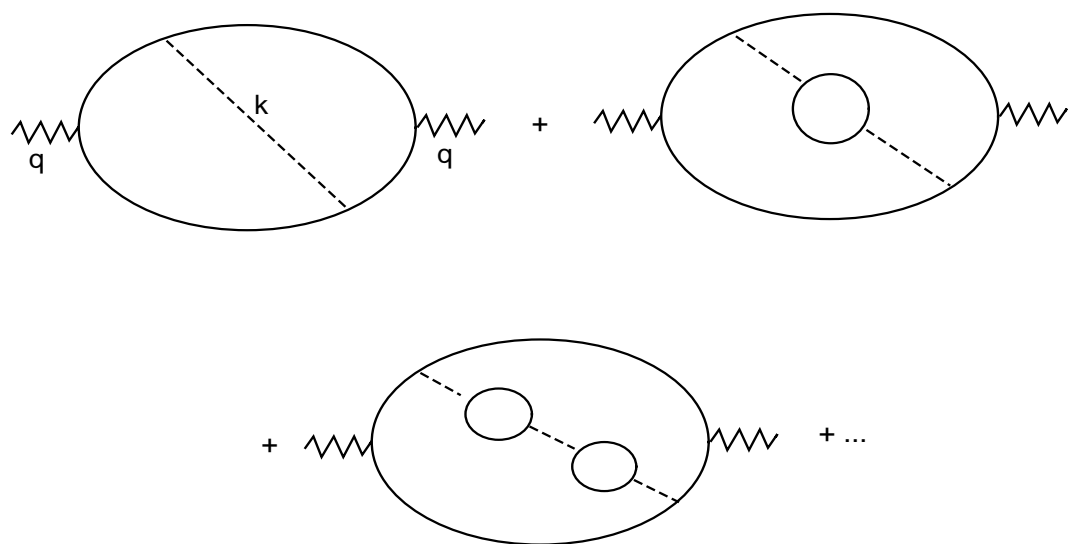


Fig. 1

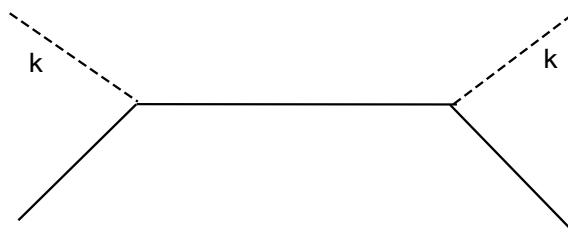


Fig. 2

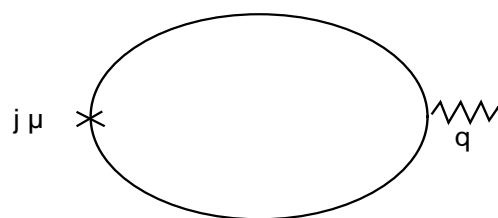


Fig. 3

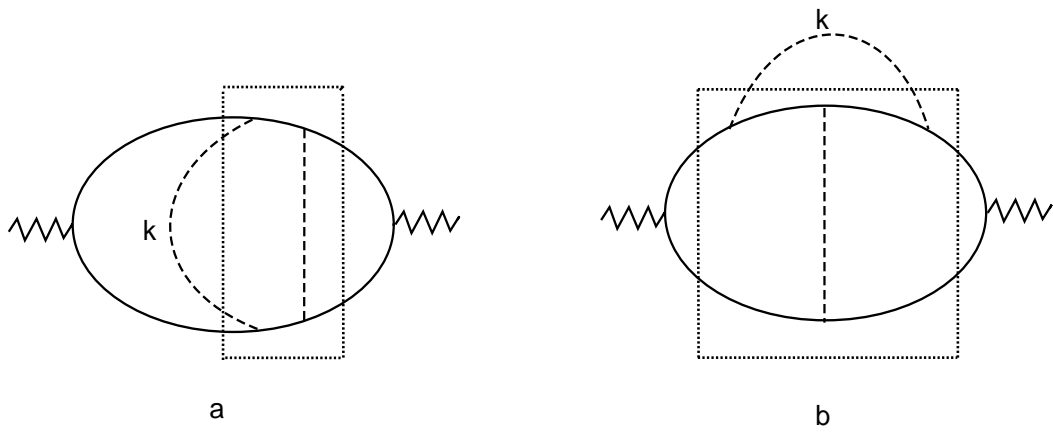


Fig. 4

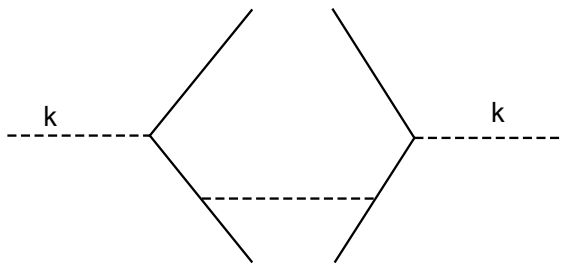


Fig. 5